

# Limit of Temperature Anisotropy Relaxation by Ion Cyclotron Waves: a Statistical Theory

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The ion velocity distribution resulting from ion anisotropy reduction by the growth of ion cyclotron waves has been investigated by the use of a Maximum Entropy Principle. The anisotropy is reduced due to waves induced by an electromagnetic instability, but the system saturates before a completely isotropic Maxwellian distribution is realized, and a nonzero level of anisotropy remains in the final state. This is because the scattering process has a restriction: waves do not alter particle energy in the wave rest frame. Taking this restriction into account as an additional constraint, the final state is calculated by maximizing the entropy. The results show that the upper bound of the plasma  $\beta$  and anisotropy are inversely correlated, which is quantitatively in agreement with recent studies on satellite observations, numerical simulations, and laboratory experiments.

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## I. INTRODUCTION

Anisotropy relaxation in collisionless plasmas proceeds in a fundamentally different way than in collisional plasmas. In the latter case, particle-particle interactions act to completely eliminate an anisotropy in the time asymptotic limit. In collisionless plasmas, wave-particle interactions also scatter particles toward the equilibrium state; however, the scattering process has some restrictions, and the waves cannot drive the plasma to a fully isotropic distribution. One good example is temperature anisotropy often found in space or laboratory plasmas.

If there is no mechanism for energy transfer between parallel and perpendicular (words parallel/perpendicular are used with respect to the ambient magnetic field throughout this paper) motion of particles, the parallel and perpendicular temperatures are completely independent; the anisotropy can be any value depending on the initial condition. In the opposite limit, that is, the perpendicular/parallel energy transfer mechanism is very efficient, the plasma would be isotropized within a sufficiently short time. Anisotropy often found in collisionless plasmas is something in-between these two extreme cases. The plasma waves scatter particles towards the equilibrium state; however, the scattering process has some restrictions, and the waves cannot drive the plasma to a complete isotropic distribution.

In this paper we investigate the relaxation of ion anisotropy by ion cyclotron waves. Recent studies in numerical simulation,<sup>1</sup> satellite observations,<sup>2</sup> and laboratory experiments<sup>3</sup> reveal that the upper bound of the ion anisotropy shows inverse correlation to the plasma  $\beta$  (ratio of the plasma pressure to the magnetic pressure) in the scattering process by ion cyclotron waves. These studies suggest that the upper bound of the anisotropy is well approximated by

$$\frac{T_{\perp}}{T_{\parallel}} - 1 = A\beta_{\parallel}^{-\alpha} \quad (1)$$

where  $T_{\perp}$  and  $T_{\parallel}$  are the perpendicular and parallel temperature of ions, and  $\beta_{\parallel}$  is the plasma  $\beta$  calculated from the ion parallel temperature;  $A$  and  $\alpha$  are empirical constants typically  $A = 0.4 \sim 1.5$  and  $\alpha = 0.4 \sim 0.7$ .

This inverse correlation has been explained by the linear theory; the upper bound corresponds to the instability threshold.<sup>4,5</sup> Above this level, wave instability takes place and reduces the isotropy. The purpose of the present study is to present another theoretical approach to this problem. This approach is complementary to the linear threshold analysis, and gives insight to the non-linear stage of the anisotropy relaxation process.

What we apply is a method of statistical mechanics: "Maximum Entropy Principle" proposed by Jaynes.<sup>6</sup> The whole theory in this paper is based on the assumption that the Maximum Entropy Principle is applicable to the coarse-grained distribution function of collisionless plasmas. The validity of the assumption is verified when we find the result agrees with the phenomena observed in real plasmas.

According to Maximum Entropy Principle, we would obtain an isotropic Maxwellian distribution as the final state if the constraints were the conservation laws of energy and mass only. However, the particle scattering process by ion cyclotron waves has an additional constraint: the energy of a particle remains constant in the wave rest frame. With this additional constraint the anisotropy/ $\beta$  inverse correlation in the form of Eq. (1) is derived.

## II. MAXIMUM ENTROPY PRINCIPLE APPLIED FOR PARTICLE SCATTERING

Suppose a plasma in a uniform magnetic field. Ions have temperature anisotropy that causes instability of ion cyclotron waves. In the following we consider parallel propagating waves ( $k_{\perp} = 0$ ) since they have the largest growth rate.<sup>7</sup>

Particle scattering by parallel propagating electromagnetic waves has a restriction such that particle kinetic energy does not change when scaled in the wave rest frame. Since the temporal variation vanishes in the wave rest frame, there is no induced electric field ( $\text{rot}\mathbf{E} = -\partial\mathbf{B}/\partial t = 0$ ). Energy of a particle does not vary in the absence of an electric field, thus the particle motion in the velocity space is confined on a sphere centered at the wave phase speed (see Figure 1). In addition, the resonance condition for  $v_{\parallel} = 0$  is  $k_{\parallel} = \infty$ , thus particles cannot move across the plane of  $v_{\parallel} = 0$ . Therefore the motion of a particle with  $v_{\parallel} < 0$  is restricted on a shell (thick line in Figure 1) specified as

$$v_{\perp}^2 + (v_{\parallel} - V_p)^2 = \text{constant}; \quad v_{\parallel} < 0, \quad (2)$$

where  $V_p = \omega/k$  is the wave phase speed. We assume that  $V_p$  does not depend on  $k$ , which is a reasonable approximation for ion cyclotron waves. It should be noted that a particles with negative (positive)  $v_{\parallel}$  interacts only with positive (negative) propagating ion cyclotron waves.<sup>8</sup>

We introduce spherical coordinates  $(v_r, \theta)$  with the origin at  $(v_{\parallel}, v_{\perp}) = (V_p, 0)$  such that (see Figure 1)

$$v_{\parallel} = -v_r \cos \theta + V_p, \quad v_{\perp} = v_r \sin \theta. \quad (3)$$

The condition Eq. (2) is then written as  $v_r = \text{constant}$  and  $0 < \theta < \theta_0$  where  $\theta_0 = \cos^{-1}(V_p/v_r)$ . Hereinafter we refer the shell specified above as  $\Sigma_{v_r}$ .

Now let us introduce a single particle probability distribution of an ion. The system would be fluctuating with time and space when the instability takes place, however, what we wish to know is the zero-th order ion distribution without the fluctuation. The velocity of an ion will fluctuate due to the wave as  $\mathbf{v} + \delta\mathbf{v}$  with  $\delta\mathbf{v}$  being the wave fluctuation part. We define the single particle probability  $p(\mathbf{v})$  as the probability that the ion has an average velocity  $\mathbf{v}$ , where ‘‘average’’ means time average over sufficiently long time compared to the wave period. In other words,  $p$  corresponds to the zero-th order distribution function in the absence of waves.

We introduce the entropy of ions based on this single particle probability distribution  $p$ . As discussed before, ions cannot move across the plane of  $v_{\parallel} = 0$ , thus, entropy of whole ions can be divided into two parts as  $S = S^+ + S^-$  where  $S^+$  ( $S^-$ ) is the entropy of the ions with positive (negative)  $v_{\parallel}$ . The entropy of the ions with negative  $v_{\parallel}$  can be defined as

$$\begin{aligned} S^- &= \frac{N}{2} \iiint p^-(\mathbf{v}) \ln p^-(\mathbf{v}) d\mathbf{v} \\ &= N\pi \int_0^{\infty} \int_0^{\theta_0} p^-(v_r, \theta) \ln p^-(v_r, \theta) v_r^2 \sin \theta d\theta dv_r, \end{aligned} \quad (4)$$

where  $p^-$  is the probability distribution of an ion with  $v_{\parallel} < 0$ , and  $N$  is the number of ions per unit spatial volume; the number of ions with negative  $v_{\parallel}$  is  $N/2$  from the symmetry.

The problem is to find a distribution that maximizes the entropy  $S = S^+ + S^-$  under the constraints. The constraint of energy conservation is on the ion kinetic energy  $E_i = E_i^+ + E_i^-$  [ $E_i^+$  ( $E_i^-$ ) is the kinetic energy of ions with positive (negative)  $v_{\parallel}$ ]. From the symmetry we can reduce the problem as to maximize  $S^-$  with the constraint on  $E^-$ . The calculation for the ions with  $v_{\parallel} > 0$  can be obtained just by flipping the coordinate, therefore, we concentrate on the ions with  $v_{\parallel} < 0$  and drop the superscript ‘‘-’’ in the following.

The conservation of energy may be written as<sup>9</sup>

$$\begin{aligned} &\frac{m_i N}{2} \iiint \mathbf{v}^2 p(\mathbf{v}) d\mathbf{v} \\ &= m_i N\pi \int_0^{\infty} \int_0^{\theta_0} [v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2] p(v_r, \theta) v_r^2 \sin \theta d\theta dv_r = \text{constant}. \end{aligned} \quad (5)$$

Besides the energy constraint, we have an additional constraint that an ion is confined on  $\Sigma_{v_r}$ , which is expressed as

$$\frac{N}{2} \int_{\Sigma_{v_r}} p(\mathbf{v}) d\mathbf{v} = \pi N \int_0^{\theta_0} p(v_r, \theta) \ln p(v_r, \theta) v_r^2 \sin \theta d\theta = n_0(v_r). \quad (6)$$

In the above expression  $n_0$  is defined such that  $n_0(v_r) dv_r$  is the number of ions between  $\Sigma_{v_r}$  and  $\Sigma_{v_r+dv_r}$ .

The precise meaning of the above expression is “the number of ions on  $\Sigma_{v_r}$  is constant.” It may not sound equivalent to the original statement “an ion is confined on  $\Sigma_{v_r}$ ” because there may be exchange of ions between two different shells,  $\Sigma_{v_r}$  and  $\Sigma_{v_r'}$ , as long as the number on each shell does not change. However, we understand the two statements are equivalent when we recall a fact deduced from quantum physics: ions are indistinguishable particles, and “exchange of ions” has no meaning.

We can apply the Lagrange’s method as in the standard procedure of statistical physics. Using Lagrange’s coefficients  $\alpha$  and  $\lambda$ , we seek for  $p$  that maximizes the following quantity.

$$\begin{aligned} \Theta = & N\pi \int_0^\infty \int_0^{\theta_0} p(v_r, \theta) \ln p(v_r, \theta) v_r^2 \sin \theta d\theta dv_r \\ & + \int_0^\infty \alpha(v_r) \pi N \int_0^{\theta_0} p(v_r, \theta) v_r^2 \sin \theta d\theta dv_r \\ & + \lambda \frac{\pi m_i N}{2} \int_0^\infty \int_0^{\theta_0} [(v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2)] p(v_r, \theta) v_r^2 \sin \theta d\theta dv_r. \end{aligned} \quad (7)$$

Note that  $\alpha$  is a function  $v_r$  because the constraint of Eq. (6) must be satisfied for each value of  $v_r$ .

The maximum is given by  $p$  that satisfies

$$\frac{\delta \Theta}{\delta p} = 0, \quad (8)$$

where  $\delta/\delta p$  denotes functional derivative. The result is

$$p(v_r, \theta) = C(v_r) \exp\{-\lambda' [v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2]\}, \quad (9)$$

where  $\lambda' = \pi m_i N \lambda / 2$  and  $C(v_r) = \exp[\alpha(v_r) \pi N - 1]$  is calculated from Eq. (6) as

$$\begin{aligned} C(v_r) &= \frac{n_0}{N} \left[ \pi \int_0^\infty \int_0^{\theta_0} p(v_r; \theta) v_r^2 \sin \theta d\theta dv_r \right]^{-1} \\ &= \frac{2n_0 \pi \lambda' V_p v_r}{N e^{-\lambda' (v_r - V_p)^2} - e^{-\lambda' (v_r^2 - V_p^2)}}. \end{aligned} \quad (10)$$

The parameter  $\lambda'$  is to be determined from the energy conservation, which will be done in the next section for a given initial distribution.

### III. CALCULATIONS AND RESULTS

Let us assume that there is no wave at the initial state and the ion distribution  $f_0$  is an anisotropic bi-Maxwellian distribution expressed as

$$f_0(v_\perp, v_\parallel) = \frac{m_i^{3/2} N}{T_\perp \sqrt{8\pi^3 T_\parallel}} \exp\left(-\frac{m_i v_\parallel^2}{2T_\parallel} - \frac{m_i v_\perp^2}{2T_\perp}\right), \quad (11)$$

where  $m_i$  is the ion mass, and the temperatures  $T_\perp$  and  $T_\parallel$  are scaled in the unit of energy. The number of ions on each shell is then given as

$$n_0(v_r) = 2\pi v_r^2 \int_0^{\theta_0} f_0(v_r, \theta) \sin \theta d\theta. \quad (12)$$

The instability of ion cyclotron waves is caused by the ion anisotropy, thus the wave energy is supplied from the ion kinetic energy. Neglecting the change of electron energy, we can write

$$(E_i + E_w)_{t=\infty} = (E_i)_{t=0} = \frac{1}{2}N(2T_{\perp} + T_{\parallel}). \quad (13)$$

In the above expression  $E_w$  represents the energy associated with the wave.

What we need for calculating the maximum entropy distribution are the wave phase speed  $V_p$  and the wave energy  $E_w$  at the saturation level. The wave phase speed  $V_p$  can be calculated from the linear theory; according to Gary *et al.*<sup>5</sup> it can be approximated as

$$V_p = 0.86V_A\beta_{\parallel}^{-0.1}, \quad (14)$$

where  $V_A$  is the Alfvén speed. Though  $\beta_{\parallel}$  varies during the relaxation process, the dependence of  $V_p$  on  $\beta_{\parallel}$  is not significant. Thus we assume  $\beta_{\parallel}$  does not vary during the scattering, and use its final value.

Wave saturation energy  $E_w$  can be estimated by the quasi-linear theory. Gary *et al.*<sup>10</sup> gives the following handy formula

$$\frac{E_w}{E_M} \simeq 0.04(\beta_{\parallel})_{t=0} \left[ \left( \frac{T_{\perp}}{T_{\parallel}} \right)_{t=0} - \left( \frac{T_{\perp}}{T_{\parallel}} \right)_{t=\infty} \right], \quad (15)$$

which is found to be in good agreement with the numerical experiment for  $0.1 < \beta_{\parallel} < 1$ . The final temperature ratio  $(T_{\perp}/T_{\parallel})_{t=\infty}$  is not known at the beginning of the calculation, thus we estimate  $(T_{\perp}/T_{\parallel})_{\infty} \sim 1$  since the anisotropy is significantly reduced at the final state as we will see below.

Final distribution can be obtained when we know the value of  $\lambda'$  in Eq. (9). Considering  $E_i^+ = E_i^-$ , the ion kinetic energy  $E_i$  with the distribution  $p$  in Eq. (9) is calculated as

$$\begin{aligned} E_i &= 2\pi\pi m_i N \int_0^{\infty} \int_0^{\theta_0} [v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2] \\ &\quad \times C(v_r) \exp\{-\lambda'[v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2]\} v_r^2 \sin \theta d\theta dv_r \\ &= 2\pi m_i N \int_0^{\infty} \left[ \frac{e^{2\lambda'V_p(V_p-v_r)} [1 - \lambda'(v_r^2 - V_p^2)] - 1 + \lambda'(v_r - V_p)^2}{\lambda' (e^{2\lambda'V_p(V_p-v_r)} - 1)} \right]. \end{aligned} \quad (16)$$

We numerically evaluate the integration in Eq. (12) and the  $r$  integration in the above equation. Then we can find  $\lambda'$  that satisfies Eq. (13) with Newton-Raphson method. The program used in this calculation is posted at <http://mira.bio.fpu.ac.jp/~tadas>.

When we have the value of  $\lambda'$ , parallel and perpendicular temperature is calculated as

$$\begin{aligned} T_{\parallel} &= 2\pi m_i N \int_0^{\infty} \int_0^{\theta_0} (v_r \cos \theta - V_p)^2 C(v_r) \\ &\quad \exp\{-\lambda'[v_r^2 \sin^2 \theta + (v_r \cos \theta - V_p)^2]\} v_r^2 \sin \theta d\theta dv_r \\ &= 2\pi m_i N \int_0^{\infty} \left[ \frac{e^{2\lambda'V_p(V_p-v_r)} - 1 + 2\lambda'V_p(v_r - V_p)[1 + \lambda'V_p(v_r - V_p)]}{2\lambda'^2 V_p^2 (e^{2\lambda'V_p(V_p-v_r)} - 1)} \right], \\ T_{\perp} &= \frac{1}{2} \left( \frac{2E_i}{N} - T_{\parallel} \right). \end{aligned} \quad (17)$$

Figure 2 is the summary of our calculation. We give initial conditions as plotted by hollow circles in Figure 2. Solid circles are the temperature anisotropy calculated for the corresponding final state given in Eq. (11). The solid line is an eye guide of  $T_{\perp}/T_{\parallel} - 1 = 0.35\beta_{\parallel}^{-0.4}$ . We see a clear upper bound in the form of Eq. (1).

#### IV. CONCLUDING REMARKS

Applying Maximum Entropy Principle proposed by Jaynes,<sup>6</sup> we have successfully obtained the limit of anisotropy relaxation by ion cyclotron waves. When there is temperature anisotropy of ions, instability of ion cyclotron waves takes place to reduce the anisotropy. If there were no constraint other than energy and mass conservation, the final state would be an isotropic Maxwellian distribution. The particle scattering by ion cyclotron waves, however, has a constraint that the ions are confined on the constant energy shell in the wave rest frame. Therefore, the resulting final distribution is still away from the complete equilibrium, and anisotropy remains at a certain level. Our calculation shows that the upper bound of this remaining anisotropy is inversely correlated to the plasma  $\beta$  as found in previous studies.

There are textbooks that state phenomena in collisionless plasmas are reversible processes since the entropy calculated from the exact distribution function remains constant.<sup>11</sup> This statement clearly contradicts with our experience. For instance, it is hard to imagine there is an inverse process to the anisotropy relaxation examined in this paper: starting from a turbulent state with less anisotropy, ion cyclotron waves make the anisotropy larger and decay out when achieving the maximum anisotropy. This means that the constancy of entropy defined with the exact distribution function does not mean the reversibility of the system. This situation is similar to the problem of Gibbs' entropy: it remains constant even for irreversible processes in collisional gases.<sup>12</sup>

The problem of the origin of irreversibility is a quite fundamental problem in physics for which we do not have satisfactory answer yet; it is far beyond the scope of this paper. Nevertheless, it is reasonable to assume the applicability of Maximum Entropy Principle for the coarse-grained entropy considering the empirical irreversibility of collisionless plasmas.<sup>13</sup> The present paper is based on this assumption.

There has been few attempts to explore statistical mechanics of collisionless plasmas; even the existence of thermal equilibrium is not theoretically well founded. The equilibrium state of a collisionless system was examined by Lynden-Bell<sup>14</sup> in the context of the distribution of stars in a galaxy. He claims that the equilibrium must be a superposition of Fermi distributions, which is called Lynden-Bell statistics. Recently Nakamura<sup>15</sup> has refuted this theory and shown that the equilibrium state is a single Maxwellian distribution. The results in this paper can be regarded as a good example of the applicability of such statistical theories to collisionless plasmas.

#### ACKNOWLEDGMENTS

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- <sup>8</sup> See, for example, Isenberg, P. A. and M. Martin, *J. Geophys. Res.* 101, 11005 (1996).
- <sup>9</sup> This energy conservation law is for the whole ions in the plasma frame: not to be confused with the single particle energy conservation in the wave rest frame as in Eq. (2).
- <sup>10</sup> Gary, S. P. and D. Winske, *J. Geophys. Res.* 98, 9171 (1993).
- <sup>11</sup> See, for example, Schmidt, G., *Physics of High Temperature Plasmas*, (Academic Press, New York, 1966).
- <sup>12</sup> R. C. Toleman, *The Principles of Statistical Mechanics* (Clarendon Press, Oxford, 1938).
- <sup>13</sup> For an exact (fine-grained) distribution, conservation of phase space volume can be a constraint, however, it is expected not to play a role for a coarse-grained distribution in the non-degenerate case as discussed by Nakamura.<sup>15</sup> Though this problem may require more philosophical investigation (e.g., what is the precise definition of a “exact distribution?”), we simply ignore the phase space volume conservation because we can obtain pragmatically satisfactory result.
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FIG. 1. Schematic illustration of ion scattering by ion cyclotron waves in the velocity space. The motion of an ion is confined on a shell indicated by the thick line.

FIG. 2. Anisotropy obtained from our calculation plotted against  $\beta_{\parallel}$ . Hollow circles are the initial values of anisotropy, and solid circles are those in the final state calculated based on Maximum Entropy Principle. Solid line is an eye guide of  $T_{\perp}/T_{\parallel} - 1 = 0.35\beta_{\parallel}^{-0.4}$ .

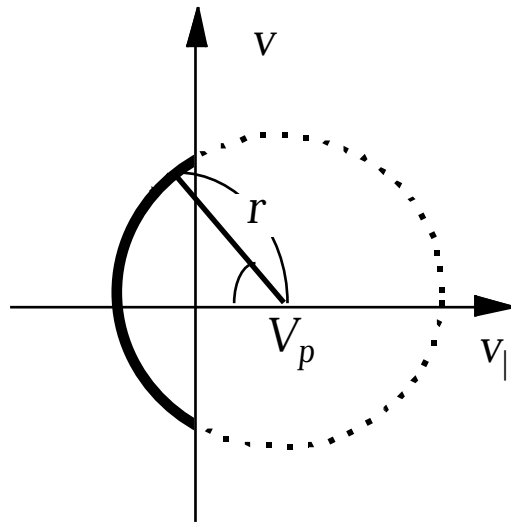


Fig 1

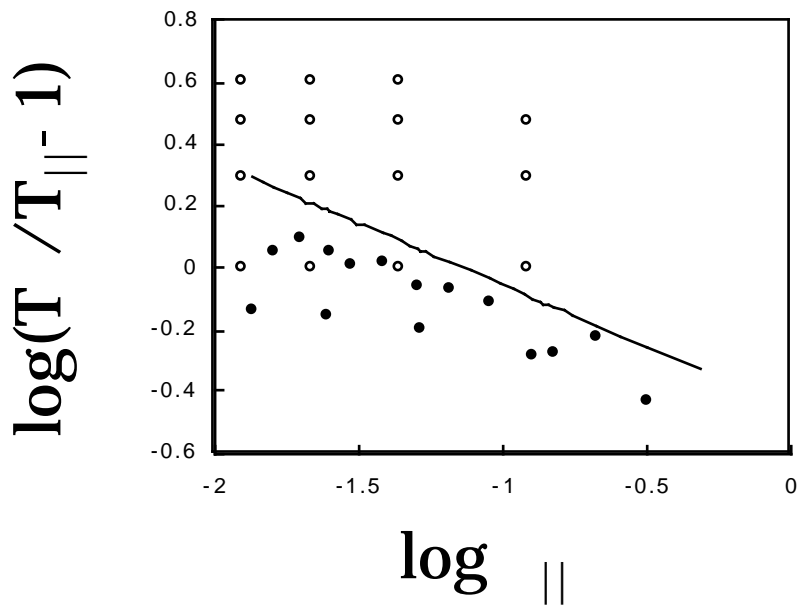


Fig 2